Problem Set 7 due April 22, at 10 PM, on Gradescope

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue.

Problem 1:

Consider the matrix $A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$ for any numbers a, b, c.

(1) Give a condition, in terms of a, b, c, for the matrix A to be invertible. *(6 points)*

(2) Assuming a, b, c satisfy the condition you found in the previous part, use the <u>cofactor formula for the inverse</u> to compute A^{-1}. *(10 points)*

(3) Use <u>Cramer's rule</u> to compute the solution to the equation:

$$A\boldsymbol{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for any numbers x, y, z. *(9 points)*

Problem 2:

Let $\boldsymbol{v} \in \mathbb{R}^n$ be a vector of length 1, and consider the matrix $A = I - 2\boldsymbol{v}\boldsymbol{v}^T$.

(1) Show that $\boldsymbol{v}$ is an eigenvector of A. What is the corresponding eigenvalue? *(5 points)*

(2) Describe $n-1$ other eigenvectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_{n-1}$ of A, which together with $\boldsymbol{v}$ form a basis of $\mathbb{R}^n$. What are the corresponding eigenvalues of these $n-1$ eigenvectors? *(10 points)*

(3) What is the rank of the matrix A? How do you know? *(5 points)*

Problem 3:

Consider the matrices:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

(1) Prove that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ by computing them explicitly. *(10 points)*

(2) Use the previous part to prove that $\mathrm{Tr}(D) = \mathrm{Tr}(VDV^{-1})$ for any 2×2 square matrix D and any invertible 2×2 matrix V. *(5 points)*

Problem 4:

Let A be an $m \times n$ matrix and B be an $n \times m$ matrix. Prove that the any <u>non-zero</u> eigenvalue of the square matrix AB is also an eigenvalue of the matrix BA. *(10 points)*

Problem 5:

Consider the matrix:

$$A = \begin{bmatrix} -3 & 4 & -8 \\ 3 & -5 & 8 \\ 2 & -3 & 5 \end{bmatrix}$$

(1) Find an eigenvalue λ of A, and compute an eigenvector $\boldsymbol{v}$. *(10 points)*

(2) Compute a vector $\boldsymbol{w}$ such that $(A - \lambda I)\boldsymbol{w} = \boldsymbol{v}$ and a vector $\boldsymbol{z}$ such that $(A - \lambda I)\boldsymbol{z} = \boldsymbol{w}$.

(10 points)

(3) Consider the matrix $V = \left[\begin{array}{c|c|c} \boldsymbol{v} & \boldsymbol{w} & \boldsymbol{z} \end{array} \right]$ and compute:

$$V^{-1}AV$$

Congratulations: you just computed the Jordan normal form of A. *(10 points)*